Is the recursive preference asset pricing model more flexible? Evidence.*

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Abstract

This paper investigates why there exists considerable variation in estimates of the coefficient of relative risk aversion (CRRA) and the elasticity of intertemporal substitution (EIS) in the consumption-based asset pricing model with Epstein and Zin (1989) preferences. Using the estimation method developed by Chen et al. (2013), we show the Epstein and Zin (1989) structure collapses to the time-separable structure. This result is consistent with the argument in Kocherlakota (1990) saying that the recursive preference-based utility function does not have more explanatory power than the time-separable one. We also show the choice of parameters might lead to "ill-behaved" conditional moment, which might cause either GMM method to get "stuck", or the estimates do not move much from the starting points. Lastly, our result is robust to the choice of instruments for computing the conditional moment function in the GMM method.

Keywords: Recursive preference consumption-based asset pricing, Monte Carlo simulation, semiparametric estimation.

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1 Literature review

Research on recursive preferences asset pricing model has been going on for decades. In this literature, the model proposed by Epstein and Zin (1989) and Weil (1989) (EZW hereafter) is widely used because of its higher degree of flexibility with regards to attitudes towards to the CRRA and the EIS. Comparing to the model with time-separable utility function such as

$$U_t = E\left[\sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\gamma}}{1-\gamma} \middle| \mathcal{F}_t\right], \tag{1.1}$$

where γ is both the CRRA and the reciprocal of the EIS. The EZW model has a recursive preferences utility function defined as

$$V_t = \left[(1-\beta)C_t^{1-\rho} + \beta(E\left[V_{t+1}^{1-\theta}|\mathcal{F}_t\right])^{\frac{1-\rho}{1-\theta}}\right]^{\frac{1}{1-\rho}}, \tag{1.2}$$

where the CRRA is characterized by θ and agent's EIS is characterized by $\frac{1}{\rho}$. It's easy to show (1.1) is a special case of (1.2) if we let $\theta = \rho$.

Because of its higher generality, the EZW model is widely used in the empirical study as researchers are interested in estimating the CRRA and the EIS separately. The estimates of those parameters, however, vary widely, and there is still no one widely accepted estimate. Epstein and Zin (1991) attribute this issue to the model's sensitivity to the choice of consumption measure and instrumental variables. Kocherlakota (1990) provides a theoretical proof that, assuming an i.i.d. endowment process, (1.1) and (1.2) are "observationally equivalent". In other words, using assets returns and aggregate consumption growth data, the empirical investigator cannot disentangle the EIS from CRRA in the EZW setting. Smith (1999) studies the finite sample properties of tests of the EZW model and points out that those tests often have little power to reject the null hypothesis that $\theta = \rho$.

In this paper, we investigate why the estimates of the CRRA and the EIS vary significantly in the literature. We use a GMM type proxy-free method to estimate the parameters, and two sources of data: empirical data and simulated data to verify the results. In both cases, we find that the surface of the GMM objective function is not globally concave, which might lead the most optimization algorithms to fail. More importantly, most local minimum points fall onto the line of $\theta = \rho$. This is a direct evidence to support the point in Kocherlakota (1990) and Smith (1999). Secondly, it may cause infinite or undefined GMM moment and lead to lead to either the estimates do not move from the starting point or no estimates are available if we don't restrict the relationship between the CRRA and the EIS. Lastly, our result shows the choice of instruments is not critical to varying estimates. The rest of the paper is organized as follows. Section 2 introduces the empirical data we use, and the data generation process of simulation. Section 3 describes the estimation method and reports the results. Section 4 concludes the paper.

2 Data

2.1 Empirical data

The empirical data we use is from Chen et al. (2013) (CFL hereafter). We provide a brief introduction here, but refer readers to CFL for the detailed description. The aggregate data are quarterly returns and consumption growth from the first quarter of 1952 to the first quarter of 2015. The data of asset returns includes 3-month Treasury bill rate, and six value-weighted portfolios of common stock. For the instruments, we choose the same variables as CFL and keep the notation consistent. The relative bill rate and the excess return on the S&P 500 index are denoted as $RREL$ and $SPEX$, respectively. The consumption-wealth ratio is measured as the cointegrating residual between log consumption, log asset wealth, and log labor income, and is denoted as $\widehat{cay}$.

2.2 Simulated data

Time series of the state variables (λ_t, ξ_t): Our data generation process is based on Kocherlakota (1990) and Smith (1999). There are two state variables: annual aggregate consumption growth ($\lambda_t = c_t/c_{t-1}$), and annual aggregate dividend growth ($\xi_t = d_t/d_{t-1}$), which are generated using the following VAR(2) model

$$\begin{aligned} \ln \lambda_t &= 0.021 + 0.017 \ln \xi_{t-1} - 0.161 \ln \lambda_{t-1} + \varepsilon_t^1, \\ \ln \xi_t &= 0.004 + 0.117 \ln \lambda_{t-1} + 0.414 \ln \xi_{t-1} + \varepsilon_t^2, \\ \Sigma(\varepsilon) &= \begin{bmatrix} 0.01400 & 0.00177 \\ 0.00177 & 0.00120 \end{bmatrix}. \end{aligned} \tag{2.1}$$

The error terms $\varepsilon = (\varepsilon^1, \varepsilon^2)$ are assumed to be jointly normally distributed with the covariance matrix $\Sigma(\varepsilon)$. The parameters are calibrated from the data of annual per-capita, real non-durable consumption growth and the annual S&P 500 aggregate, real dividend growth over the period 1888-1978.

Transition matrix from Tauchen's method: We use Tauchen's quadrature method in Tauchen and Hussey (1991) to discretize the continuous stochastic process of $\{\lambda_t,\ \xi_t : t = 1, \cdots, T\}$ to a finite Markov-chain with state values of $\{\lambda_i,\ \xi_j : i = 1, \cdots, S, j = 1, \cdots, S\}$ and a transition matrix $\Pi = \{\pi_{ij} = Pr(s_{t+1} = \lambda_j, \xi_j | s_t = \lambda_i, \xi_i) : i = 1, \cdots, S, j = 1, \cdots, S\}$, where S is the number of states.

Three simulated returns: Given the state value of (λ, ξ) and the $S \times S$ transition matrix Π, along with the parameter set (β, θ, ρ), we can then simulate the return on aggregate wealth (R_w), the return on risky asset (R_{sp}), and the risk-free rate (R_f) (which is defined as the return on a claim paying a unit of consumption for sure one period forward) as follows. Let us consider a simple Lucas-style asset pricing model case in which the return is the future price of asset plus the dividend income. Denote $p_{w,t}$, and $d_{w,t}$ to be the price and dividends of the aggregate wealth portfolio in period t, respectively. We assume the representative agent receives only dividends as income and consumes all dividends when received, i.e., $d_{w,t}/d_{w,t-1} = \lambda_t$, which is the same as in Smith (1999). Then the finite-states Markov representation of the Euler equation takes the form of

$$\begin{aligned} p_{w,i} &= \sum_{j=1}^{S} \pi_{ij}[(p_{w,j} + d_{w,j})m_{ij}(c_i, c_j)], \\ m_{ij} &= \left(\beta \left(\frac{c_j}{c_i}\right)^{-\rho}\right)^{\frac{1-\theta}{1-\rho}} (\mathcal{R}_{w,ij})^{\frac{1-\theta}{1-\rho}-1}, \end{aligned} \tag{2.2}$$

where m_{ij} is the stochastic discount factor. Let $v_{w,t} = p_{w,t}/d_{w,t}$ be the price-dividend ratio of the aggregate wealth portfolio. With previous assumption that dividend growth is equal to the consumption growth, equation (2.2) can be written as

$$v_{w,i}^{(1-\theta)/(1-\rho)} = \sum_{j=1}^{S} \pi_{ij}[\beta^{(1-\theta)/(1-\rho)} \lambda_j^{1-\theta} (1 + v_{w,j})^{(1-\theta)/(1-\rho)}], \quad i = 1, 2, ..., S. \tag{2.3}$$

Equation (2.3) comprises of a system of S linear equations in $v_{w,i}$ which can be solve directly. Let v_w^* be the vector of solutions to (2.3) given (β, θ, ρ). The state values of returns can be represented as the function of v_w^*. The return on aggregate wealth over state i and j is given by

$$R_{w,ij} = \frac{1 + v_{w,j}^*}{v_{w,i}^*} \lambda_j. \tag{2.4}$$

The return on risk-free asset, which is defined as the price in state i of a bond paying sure unity next period is

$$\frac{1}{R_{fi}} = \beta^{(1-\theta)(1-\rho)} \sum_{j=1}^{S} \pi_{ij} \lambda_j^{-\theta} \left(\frac{1 + v_{w,j}^*}{v_{w,i}^*}\right)^{(\rho-\theta)(1-\rho)}. \tag{2.5}$$

For the return on risky asset, everything is the same as return on aggregate wealth except that consumption growth is replaced by the simulated S&P 500 dividend stream. The state returns are given by

$$R_{sp,ij} = \frac{1 + v^*_{w,j}}{v^*_{w,i}} \xi_j. \tag{2.6}$$

Once we obtain the vector of state value of three returns, we then use the following algorithm to simulate the time-series of each return:

(1) Draw a random variable u_0 from a uniform distribution $U(0, 1)$. Let the initial state, n_0, be the smallest number such that $p(1) + p(2) + \cdots + p(n_0) \geq u_0$, where $p(i)$ is the stationary probability of being in state i, $1 \leq n_0 \leq S$.

(2) Let n' be the current state and n'' be the next state to be drawn. Draw u'' from $U(0, 1)$ and let the n'' be the smallest number such that $\sum_i^{n'} \pi(n', i) \geq u''$, where $\pi(i, j)$ is the transition probability from state i to state j.

(3) Set $n' = n''$ and then return to step 2 until $t = T$. The time series of three returns are obtained by choosing $R_w(n', n'')$, $R_f(n', n'')$, and $R_{sp}(n', n'')$.

In the literature, the constant discount factor β is not very controversial. In order to reduce the computation burden, we fix it at 0.99, which is a standard value in the literature. The estimates of CRRA (θ) and EIS ($\frac{1}{\rho}$) vary widely. Nestor and Ruben (2015) show that the most widely accepted measures of θ lie between 1 and 3, and ρ is widely agreed to be less than 1. Following Smith (1999), we set the true values of the parameters in the simulation to be $(\theta, \rho) \in \{(0.8, 0.8), (0.8, 1.3), (1.3, 1.3), (1.3, 5.2)\}$. The mean and standard deviation of the simulated state variables and returns are reported in Table (2.1). Across all combinations of parameters, the return on the aggregate wealth portfolio (R_w) and the return on risky asset (R_{sp}) have higher return and risk compared to the risk-free asset, but the equity premium ($R_{sp} - R_f$) doesn't vary much. This result provides an evidence for Kocherlakota

(1990) arguing that separating the parameters of CRRA and EIS does not solve the equity premium puzzle.

Table 2.1: Summary information on state variables and simulated returns from Monte Carlo simulations using the parameter values from Epstein and Zin (1991).

θ, ρ		λ_t	ξ_t	R_w	R_f	R_{sp}	Equity premium
0.80, 0.80	Estimator	1.026	1.007	1.023	1.021	1.031	0.010
	Std.dev.	0.108	0.107	0.128	0.015	0.138	
0.80, 5.20	Estimator	1.030	1.042	1.065	1.028	1.029	0.000
	Std.dev.	0.111	0.142	0.133	0.027	0.133	
1.35, 1.35	Estimator	1.000	1.024	1.042	1.020	1.039	0.019
	Std.dev.	0.094	0.129	0.128	0.025	0.134	
1.35, 5.20	Estimator	1.036	0.988	1.111	1.086	1.116	0.030
	Std.dev.	0.125	0.121	0.188	0.101	0.150	

3 Estimation and results

The estimation method: We use a two-steps semi-parametric estimation approach developed by CFL to estimate (β, θ, ρ). This approach directly estimates the unobservable value function without requiring the proxy for $\mathcal{R}_w$. We provide a brief introduction to CFL's method here. For the large sample properties and proof, see Ai and Chen (2003). Recall that the EZW utility function is defined recursively by

$$\begin{aligned} V_t &= \left[(1-\beta)C_t^{1-\rho} + \beta\mathcal{R}_t(V_{t+1})^{1-\rho}\right]^{\frac{1}{1-\rho}}, \\ \mathcal{R}_t\left(V_{t+1}\right) &\equiv \left(E\left[(V_{t+1})^{1-\theta}|\mathcal{F}_t\right]\right)^{\frac{1}{1-\theta}}, \end{aligned} \tag{3.1}$$

where $\mathcal{R}_t(V_{t+1})$ is the risk adjustment to the date $t+1$ continuation value function. Dividing both sides by C_t, we have

$$\frac{V_t}{C_t} = \left[(1-\beta) + \beta\mathcal{R}_t\left(\frac{V_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\right)^{1-\rho}\right]^{\frac{1}{1-\rho}}. \tag{3.2}$$

The stochastic discount factor (SDF) is

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t}}{\mathcal{R}_t \left(\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t} \right)} \right)^{\rho - \theta} . \tag{3.3}$$

Rearranging (3.2) and plugging it into (3.3), the SDF becomes

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{\frac{V_{t+1}}{C_{t+1}} \frac{C_{t+1}}{C_t}}{\left(\frac{1}{\beta} \left[\left(\frac{V_t}{C_t} \right)^{1-\rho} - (1 - \beta) \right] \right)^{\frac{1}{1-\rho}}} \right)^{\rho - \theta} . \tag{3.4}$$

The only latent variable in (3.4) is the continuation value function-to-consumption ratio, $\frac{V_t}{C_t}$. CFL estimate $\frac{V_t}{C_t}$ by a sequence of flexible parametric functions, with the number of parameters expanding as the sample size grows. To make it simple, the $\frac{V_t}{C_t}$ is approximated by a linear combination of basis functions, such as

$$\frac{V_t}{C_t} \approx F_t(\cdot, \delta) = a_0(\delta) + \sum_{j=1}^{K_T} a_j(\delta) B_j \left(\frac{V_{t-1}}{C_{t-1}}, \frac{C_t}{C_{t-1}} \right), \tag{3.5}$$

where the coefficients $\{a_0, a_1, ..., a_{K_T}\}$ depend on $\delta = (\beta, \theta, \rho)$. The basis functions $\{B_j(\cdot, \cdot) : j = 1, ...K_T\}$ with the dimensionality of K_T have known functional forms, and are independent of δ. We use the following B-splines with degree $m = 3$ as our basis function:

$$B_m(y) = \frac{1}{(m-1)!} \sum_{k=0}^{m} (-1)^k \begin{pmatrix} m \\ k \end{pmatrix} [\max(0, y - k)]^{m-1}. \tag{3.6}$$

From now on, the latent value function is completely characterized by the parameters $\{\frac{V_0}{C_0}, a_0, a_1, ..., a_{K_T}\}$.

The estimation is based on the following two-steps GMM method. Given the initial value of $\delta = (\beta, \theta, \rho)$, we first estimate $\hat{F}_t(\cdot, \delta)$ by minimizing

$$\hat{F}_t(\cdot, \delta) = \underset{F_{K_T} \in \mathcal{V}_T}{\operatorname{argmin}} \left[g_t(\delta, F_t(\delta, \cdot))\right]' W_t \left[g_t(\delta, F_t(\delta, \cdot))\right], \tag{3.7}$$

where g_t is a $N \times 1$ vector of $g_{i,t}$ defined by

$$g_{i,t}(\delta, F_t) = \frac{1}{T} \sum_{t=1}^{T} \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{F_{t+1} \cdot \frac{C_{t+1}}{C_t}}{\left(\frac{1}{\beta} \left[F_t^{1-\rho} - (1-\beta) \right] \right)^{\frac{1}{1-\rho}}} \right)^{\rho - \theta} R_{i,t+1} - 1 \right] \otimes x_t, \tag{3.8}$$

where $\otimes$ is the kronecker product, and x_t is the state variable that captures the information in period t. For the estimation using simulated data, we choose the lag of consumption growth and dividend growth as x_t as they are the "true value" in our simulation. For the empirical data in CFL, we provide the result using the same instruments as in CFL as well as without any instrument. The details will be discussed later in this section. With the optimal $\hat{F}_t$ in hand, the estimators of δ are defined as

$$\hat{\delta} = \underset{\delta \in \mathcal{D}}{\operatorname{argmin}} \left[g_t \left(\delta, \hat{F}_t(\delta) \right) \right]' W_t \left[g_t \left(\delta, \hat{F}_t(\delta) \right) \right], \tag{3.9}$$

where $\mathcal{D}$ is the parameter set. The two steps are repeated until the convergence condition is satisfied.

Estimation using the empirical data. Instead of providing a point estimator of (θ, ρ) in the EZW model, we believe showing the surface of the GMM objective function can give readers a better idea why the estimation of these parameters is controversial in the literature. In Figure 3.1, the left panel is the surface of GMM object using instruments. We choose $x_t = \left(c\hat{a}y_t, RREL_t, SPEX_t, \frac{C_t}{C_{t-1}} \right)$, which is consistent with CFL.

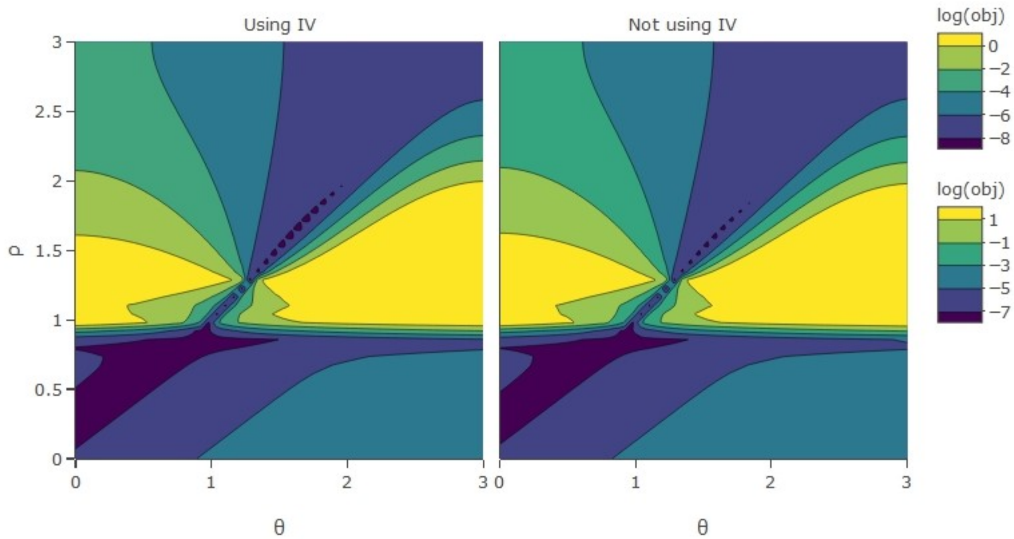

Figure 3.1: Surface of GMM objective function using data of CFL

In the right panel, we let x_t be identity matrix. As shown, the state variables x_t does not affect the surface too much. In both panels, the surface is not globally concave, and this may cause most of the optimization algorithm to fail. A common solution to this issue is to start with many different initial values. However, this may cause the estimators to heavily depend on the choice of the starting points, especially in the nonlinear case. We conjecture that it is the main reason why the estimation of CRRA and EIS vary widely in the literature. Another important finding is that most local minimum of GMM objective function fall onto the line of $\theta = \rho$, in which case the recursive preference model degenerates to the time-separable model. It is a direct numerical evidence to support the point in Kocherlakota (1990) that though the EZW model is a more general setting than the time-separable model, it has no more explanatory power than the latter one. Last but not least, some combinations of the parameters may lead to the ill-behaved objective function or generate "infinite moment" as it is pointed out in Tauchen (1986). This partially explains why the GMM method is frequently "stuck" during the estimation procedure.

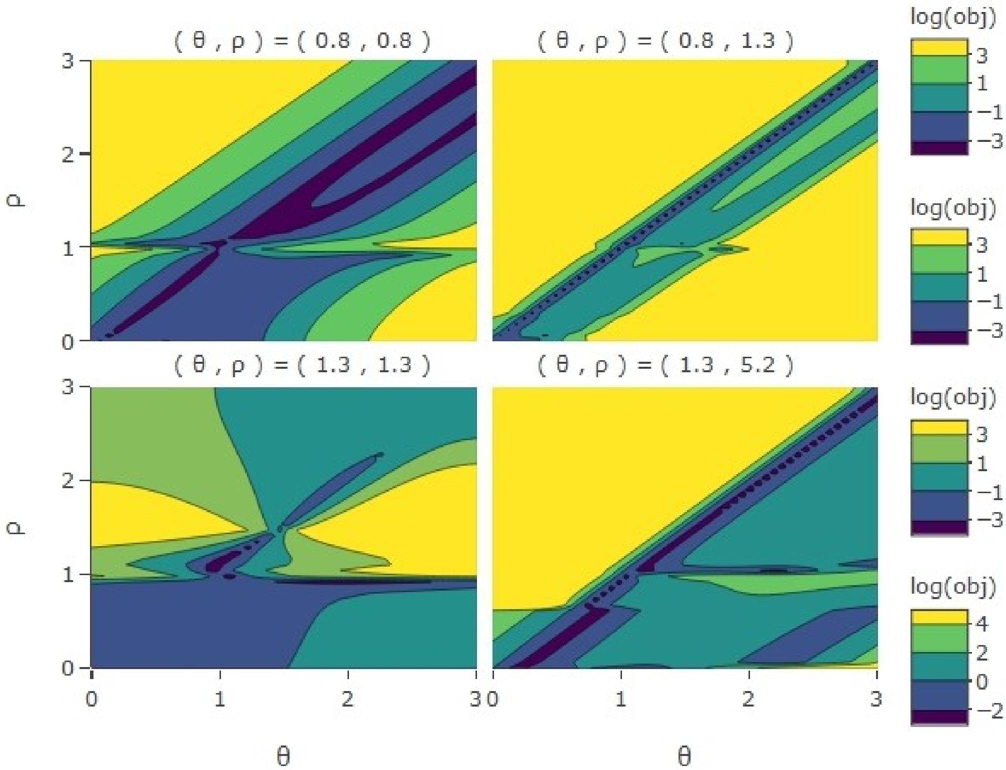

Figure 3.2: Surface of GMM objective function using simulated data

Estimation using the simulated data. To verify our results, we run a Monte Carlo simulation with true values of $(\theta, \rho) \in \{(0.8, 0.8), (0.8, 1.3), (1.3, 1.3), (1.3, 5.2)\}$. Using the same GMM estimation method and letting the state variable be $x_t = (\lambda_{t-1}, \xi_{t-1})$, we report the surface of GMM objective function in Figure 3.2. Firstly, similar results as in the case using CFL's data show up across all combinations of parameters. In each panel of figure 3.2, most of the local minimum points fall onto the line of $\rho = \theta$ even if the true value of ρ, θ are different. This confirms that the empirical investigators with data on asset prices and aggregate consumption cannot disentangle CRRA form EIS. Secondly, for example in the case of $(\theta, \rho) = (0.8, 1.3)$, the frequency of "infinite moment" is very high, which may cause either the GMM method get "stuck" at most initial points, or the estimator does not move from where it starts. Both are commonly encountered problems in the literature.

4 Conclusion

This paper is an attempt to numerically investigate "identifiability" of the CRRA and EIS in the EZW model. We find that most of the local minima fall onto the line of $\theta = \rho$ implying that optimization routines cannot disentangle θ from ρ. This result implies that asset pricing model with recursive preference does not have more explanatory power than the model with time-separable utility. Secondly, we find that the GMM moment condition in the case of the EZW model is likely to be "infinite" or undefined unless we impose a substantial restriction on the relation between the CRRA and the EIS. These numerical issues might cause either GMM method to fail or the estimates to remain unchanged from the initial values. Lastly, our first finding is robust to the choice of instrumental variables.

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